DETERMINING INCLUSIONS FOR THE MAXWELL’S EQUATIONS

1. Enclosing obstacle

1.1. Direct Problems and CGO-solutions. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with a \( C^{2,1} \)-boundary and a connected complement \( \mathbb{R}^3 \setminus \bar{\Omega} \). Assume \( D \subset \Omega \) is the obstacle or cavity. The electric permittivity \( \varepsilon_0 \), conductivity \( \sigma_0 \) and magnetic permeability \( \mu_0 \) have the following properties: there are positive constants \( \varepsilon_m, \varepsilon_M, \mu_m, \mu_M \) and \( \sigma_M \) such that for all \( x \in \Omega \)

\[
\varepsilon_m \leq \varepsilon_0(x) \leq \varepsilon_M, \quad \mu_m \leq \mu_0(x) \leq \mu_M, \quad 0 \leq \sigma_0(x) \leq \sigma_M
\]

and \( \varepsilon_0 - \varepsilon_c, \sigma_0 - \sigma_c, \mu_0 - \mu_c \in C^3(\Omega) \) for positive constants \( \varepsilon_c \) and \( \mu_c \).

Considering the boundary value problem of Maxwell’s equations

\[
\nabla \wedge E = i \omega \mu_0 H, \quad \nabla \wedge H = -i \omega \gamma_0 E \quad \text{in} \ \Omega \setminus \bar{D},
\]

\[
\nu \wedge E = f \in TH^{1/2} \left( \partial \Omega \right) \quad \text{on} \ \partial \Omega,
\]

(1.1)

where \( \gamma_0 = \varepsilon_0 + i \frac{\sigma_0}{\omega} \), and zero tangential magnetic field condition on \( \partial D \)

\[
(\nu \wedge H)|_{\partial D} = 0,
\]

(1.2)

here \( \nu \) is the unit outer normal vector on \( \partial D \). Through this note, we assume the non-dissipative case \( \sigma_0 = 0 \). Then (1.1) degenerates to

\[
\nabla \wedge (\mu_0^{-1} \nabla \wedge E) = \omega^2 \varepsilon_0 E \quad \text{in} \ \Omega \setminus \bar{D}.
\]

(1.3)

Notations. If \( F \) is a function space on \( \partial \Omega \), the subspace of all those \( f \in F^3 \) which are tangent to \( \partial \Omega \) (orthogonal to the exterior unit normal vector field of \( \partial \Omega \)) is denoted by \( TF \). For example, for \( u \in (H^s(\partial \Omega))^3 \ (s \leq 2) \), we have the decomposition \( u = u_t + u_\nu \nu \), where the tangential component \( u_t = -\nu \wedge (\nu \wedge u) \in TH^s(\partial \Omega) \) and the normal component \( u_\nu = u \cdot \nu \in H^s(\partial \Omega) \). Therefore, we have a decomposition of space \( H^s(\partial \Omega)^3 = TH^s(\partial \Omega) \oplus H^s(\partial \Omega) \). For a bounded domain \( \Omega \) in \( \mathbb{R}^3 \), we denote

\[
TH^{1/2} \left( \partial \Omega \right) = \{ u \in H^{1/2}(\partial \Omega)^3 \mid \text{Div}(u) \in H^{1/2}(\partial \Omega) \},
\]

\[
H^1 \left( \partial \Omega \right) = \{ u \in H^1(\Omega)^3 \mid \text{Div}(\nu \wedge u) \in H^{1/2}(\partial \Omega) \},
\]

with norms

\[
\| u \|^2_{TH^{1/2} \left( \partial \Omega \right)} = \| u \|^2_{H^{1/2}(\partial \Omega)^3} + \| \text{Div}(u) \|^2_{H^{1/2}(\partial \Omega)},
\]

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In addition, we define the weighted $L^2$-norm with norm $(\Omega)$. We also have the Hilbert space

$$H(\nabla \wedge, \Omega) = \{ u \in L^2(\Omega) \mid \nabla \wedge u \in L^2(\Omega)^3 \}$$

with norm

$$\|u\|_{H(\nabla \wedge, \Omega)}^2 = \|u\|_{L^2(\Omega)^3}^2 + \|\nabla \wedge u\|_{L^2(\Omega)^3}^2.$$ 

In addition, we define the weighted $L^2$ space in $\mathbb{R}^3$:

$$L^2_\delta = \left\{ f \in L^2_{loc}(\mathbb{R}^3) : \|f\|_{L^2_\delta}^2 = \int (1 + |x|^2) \|f(x)\|^2 dx < \infty \right\}.$$ 

**Admissibility.** It can be shown that for $f \in TH^{1/2}(\partial \Omega)$ and $g \in TH^{-1/2}(\partial D)$, the boundary value problem of Maxwell’s equations

$$\begin{cases}
\nabla \wedge E = i\omega \mu_0 H, \quad \nabla \wedge H = -i\omega \gamma_0 E & \text{in } \Omega \setminus \bar{D}, \\
u \wedge E|_{\partial \Omega} = f \\
u \wedge H|_{\partial D} = g,
\end{cases} \quad (1.4)$$

has a unique solution $(E, H) \in H(\nabla \wedge, \Omega \setminus \bar{D}) \times H(\nabla \wedge, \Omega \setminus \bar{D})$, except for a discrete set of magnetic resonance frequencies $\{\omega_n\}$. A proof for Dirichlet problem can be found in [9]. Moreover, we have the continuous dependency of the $H(\nabla \wedge)$-norm of the solution on the boundary condition

$$\begin{align*}
\|E\|_{H(\nabla \wedge, \Omega \setminus \bar{D})} &\leq C(\|f\|_{H^{1/2}(\partial \Omega)} + \|g\|_{H^{-1/2}(\partial D)}), \\
\|H\|_{H(\nabla \wedge, \Omega \setminus \bar{D})} &\leq C(\|f\|_{H^{1/2}(\partial \Omega)} + \|g\|_{H^{-1/2}(\partial D)}). \quad (1.5)
\end{align*}$$

At the same time, a similar proof as in [6] shows that the BVP (1.4) is well posed for $f \in TH^{1/2}_{\text{div}}(\partial \Omega)$, and $g \in TH^{1/2}_{\text{div}}(\partial D)$, i.e., except for the resonance frequencies there exists a unique solution $(E, H) \in H^{1/2}_{\text{div}}(\Omega \setminus \bar{D}) \times H^{1/2}_{\text{div}}(\Omega \setminus \bar{D})$ s.t.,

$$\begin{align*}
\|E\|_{H^{1/2}_{\text{div}}(\Omega \setminus \bar{D})} &\leq C(\|f\|_{TH^{1/2}_{\text{div}}(\partial \Omega)} + \|g\|_{TH^{1/2}_{\text{div}}(\partial D)}), \\
\|H\|_{H^{1/2}_{\text{div}}(\Omega \setminus \bar{D})} &\leq C(\|f\|_{TH^{1/2}_{\text{div}}(\partial \Omega)} + \|g\|_{TH^{1/2}_{\text{div}}(\partial D)}).
\end{align*}$$

Let $(E_0, H_0)$ denotes the solution without the obstacle.

With well-posedness of the direct problem, the impedance map

$$\Lambda_D(\nu \wedge E|_{\partial \Omega}) = \nu \wedge H|_{\partial \Omega},$$

where $\nu$ is the unit outer normal on $\partial \Omega$, is bounded from $TH^{1/2}(\partial \Omega)$ to $TH^{-1/2}(\partial \Omega)$ ([9]). Moreover, it is an isomorphism from $TH^{1/2}_{\text{div}}(\partial \Omega)$ to $TH^{1/2}_{\text{div}}(\partial \Omega)$, see [6].

The reconstruction of the obstacle will use the CGO-solution constructed in [5].

**CGO-solution** In [5], the Maxwell’s equation was reduced to an $8 \times 8$ second
order Schrödinger vector equation by introducing the generalized Sommerfeld potential. A vector CGO-solution (Sommerfeld potential) was constructed for the Schrödinger equation, and the proof of the uniqueness is facilitated compared to [6]. Same technique also appears in dealing with the inverse boundary value problems for Maxwell’s equations with partial data [2]. The construction is in $\mathbb{R}^3$.

Define the scalar fields $\Phi$ and $\Psi$ as

$$\Phi = \frac{i}{\omega} \nabla \cdot (\gamma \mu_0 E), \quad \Psi = \frac{i}{\omega} \nabla \cdot (\mu_0 H).$$

Under some assumptions on $\Phi$ and $\Psi$, we have Maxwell’s equation is equivalent to

$$\nabla \wedge E - \frac{1}{\gamma} \nabla \left( \frac{1}{\mu} \Psi \right) - i \omega \mu H = 0, \quad \nabla \wedge H + \frac{1}{\mu} \nabla \left( \frac{1}{\gamma} \Phi \right) + i \omega \gamma E = 0.$$

Moreover, in this case, $\Phi$ and $\Psi$ vanish. Let

$$X = (\phi, e, h, \psi)^T \in (\mathcal{D}')^8$$

with

$$e = \gamma_0^{1/2} E, \quad h = \mu_0^{1/2} H,$$

$$\phi = \frac{1}{\gamma_0 \mu_0} \frac{1}{2} \Phi, \quad \psi = \frac{1}{\gamma_0^{1/2} \mu_0} \Psi.$$

Then $X$ satisfies

$$(P(i\nabla) - k + V)X = 0, \quad \text{in } \Omega \quad (1.6)$$

where

$$P(i\nabla) = \begin{pmatrix} 0 & \nabla \cdot & 0 & 0 \\ \nabla \cdot & 0 & \nabla \wedge & 0 \\ 0 & -\nabla \wedge & 0 & \nabla \\ 0 & 0 & \nabla \cdot & 0 \end{pmatrix},$$

$$V = (k - \kappa) 1_8 + \left( \begin{pmatrix} 0 & \nabla \cdot & 0 & 0 \\ \nabla \cdot & 0 & -\nabla \wedge & 0 \\ 0 & \nabla \wedge & 0 & \nabla \\ 0 & 0 & \nabla \cdot & 0 \end{pmatrix} D \right) D^{-1},$$

$$D = \text{diag}(\mu_0^{1/2}, \gamma_0^{1/2} 1_3, \mu_0^{1/2} 1_3, \gamma_0^{1/2}), \quad \kappa = \omega (\gamma_0 \mu_0)^{1/2}, \quad k = \omega (\varepsilon_0 \mu_0)^{1/2}.$$

A desirable property of this operator is

$$(P(i\nabla) - k + V)(P(i\nabla) + k - V^T) = -(\Delta + k^2) 1_8 + Q,$$

where

$$Q = VP(i\nabla) - P(i\nabla)V^T + k(V + V^T) - VV^T$$

is a zeroth-order matrix multiplier. Based on this, by writing an ansatz for $X$, we define the generalized Sommerfeld potential $Y$

$$X = (P(i\nabla) + k - V^T)Y.$$
So it satisfies the Schrödinger equation
\[ (-\Delta - k^2 + Q)Y = 0. \quad (1.7) \]
The following CGO-solution is due to the Faddeev’s Kernel. Let \( \zeta \in \mathbb{C}^3 \) be a vector with \( \zeta \cdot \zeta = k^2 \). Suppose \( y_{0,\zeta} \in \mathbb{C}^8 \) is a constant vector with respect to \( x \) and bounded with respect to \( \zeta \), there exist a unique solution of (1.7) of the form
\[ Y_{\zeta}(x) = e^{ix \cdot \zeta}(y_{0,\zeta} - v_{\zeta}(x)), \]
where \( v_{\zeta}(x) \in (L^2_{\delta+1})^8 \), and
\[ \|v_{\zeta}\|_{(L^2_{\delta+1})^8} \leq C/|\zeta| \]
for \( \delta \in (-1, 0) \). Moreover, one can show that \( v_{\zeta} \in H^s(\Omega)^8 \) for \( 0 \leq s \leq 2 \), e.g., see [1], and
\[ \|v_{\zeta}(x)\|_{H^s(\Omega)^8} \leq C|\zeta|^{s-1}. \quad (1.8) \]
Lemma 3.1 in [5] states that if we choose \( y_{0,\zeta} \) such that the first and the last components of \( (P(-\zeta) - k)y_{0,\zeta} \) vanish, then for large \( |\zeta| \), \( X_{\zeta} \) provides the solution of the original Maxwell’s equation.

Let’s examine this \( X_{\zeta} \) more closely by giving specific choices of vectors.

\[ X_{\zeta} = e^{ix \cdot \zeta} \left[ (P(-\zeta) + k)y_{0,\zeta} + \left( P(-\zeta)u_{\zeta} + P(i\nabla)v_{\zeta} - V^T y_{0,\zeta} + k\nu_{\zeta} - V^T \nu_{\zeta} \right) \right]. \quad (1.9) \]

As in [5], we choose
\[ y_{0,\zeta} = \frac{1}{|\zeta|}(\zeta \cdot a, ka, kb, \zeta \cdot b)^T, \]
where
\[ \zeta = -i\tau \rho + \sqrt{\tau^2 + k^2} \rho^\perp, \]
with \( \rho, \rho^\perp \in S^2 \) and \( \rho \cdot \rho^\perp = 0 \). \( \tau > 0 \) is used to control the size of \( |\zeta| = \sqrt{2\tau^2 + k^2} \).

Taking \( \tau \to \infty \), we have
\[ \frac{\zeta}{|\zeta|} \to \check{\zeta} = \frac{1}{\sqrt{2}}(-i\rho + \rho^\perp). \]

Choosing \( a \) and \( b \) such that
\[ \check{\zeta} \cdot b = 1, \quad \check{\zeta} \cdot a = 0, \]
e.g., when \( n \geq 3 \), let
\[ a \perp \rho, a \perp \rho^\perp; \quad b = \frac{\check{\zeta}}{|\check{\zeta}|^2}. \]
The choice of \( y_{0,\zeta} \) is such that
\[ x_{0,\zeta} := (P(-\zeta) + k)y_{0,\zeta} = \frac{1}{|\zeta|} \begin{pmatrix} 0 \\ -\zeta \cdot a - k\zeta \wedge b + k^2 a \\ k\zeta \wedge a - (\zeta \cdot b)\zeta + k^2 b \\ 0 \end{pmatrix}. \]
It's easy to see that

\[ \eta = (x_0, \zeta)_2 \to -k \zeta \wedge b \ (\sim \mathcal{O}(1)), \]

\[ \theta = (x_0, \zeta)_3 \sim \mathcal{O}(\tau) \]
as \( \tau \to \infty \). Then \( X_\zeta \) is written in the form

\[ X_\zeta = e^{\tau(x \cdot \rho)+i\sqrt{\tau^2+k^2}x \cdot \rho}(x_0, \zeta + r_\zeta(x)) \]

where

\[ r_\zeta = P(-\zeta)v_\zeta + P(i\nabla)v_\zeta - V^T y_0, \zeta + k v_\zeta - V^T v_\zeta \]
satisfying for \( C > 0 \) independent of \( \zeta \).

\[ \|r_\zeta\|_{L_3^3(\Omega)^3} \leq C. \]

Hence the CGO solution of the Maxwell’s equation is given by

\[ E_0 = \varepsilon_0^{-1/2} e^{\tau(x \cdot \rho)+i\sqrt{\tau^2+k^2}x \cdot \rho}(\eta + R) \]

\[ H_0 = \mu_0^{-1/2} e^{\tau(x \cdot \rho)+i\sqrt{\tau^2+k^2}x \cdot \rho}(\theta + Q) \]

where \( \eta = \mathcal{O}(1), \theta = \mathcal{O}(\tau), R, Q \in L_3^3(\mathbb{R}^3) \) are bounded for \( \tau \gg 1 \).

1.2. Main result. Adding a parameter \( t \) in the weight, we use the CGO solution

\[ E_0 = \varepsilon_0^{-1/2} e^{\tau(x \cdot \rho-t)+i\sqrt{\tau^2+k^2}x \cdot \rho}(\eta + R) \]

\[ H_0 = \mu_0^{-1/2} e^{\tau(x \cdot \rho-t)+i\sqrt{\tau^2+k^2}x \cdot \rho}(\theta + Q) \]

(1.10)
to define an indicator function and a support function

**Definition 1.** Define

\[ I_\rho(\tau, t) := \int_{\partial \Omega} (\nu \wedge E_0) \cdot ((\Lambda \rho - \Lambda) (\nu \wedge E_0) \wedge \nu) \ dS \]
to be the indicator function

**Definition 2.** Define the support function of the convex hull of \( D \)

\[ h_D(\rho) := \sup_{x \in D} x \cdot \rho \]

for a fixed \( \rho \in S^2 \).

Now we are ready to state our main result.

**Theorem 1.1.** We assume that the set \( \{ x \in \mathbb{R}^3 \mid x \cdot \rho = h_D(\rho) \} \cap \partial D \) consists of one point and the Gaussian curvature of \( \partial D \) is not vanishing at that point.

Then, we can recover \( h_D(\rho) \) by

\[ h_D(\rho) = \inf\{ t \in \mathbb{R} \mid \lim_{\tau \to \infty} I_\rho(\tau, t) = 0 \}. \]

Moreover, if \( D \) is strictly convex, then we can reconstruct \( D \).
Remark 1. The proof of Theorem 1.1 mainly consists of showing the following limits:

\[ \lim_{\tau \to \infty} I_\rho(\tau, t) = 0, \quad \text{when } t > h_D(\rho); \quad (1.11) \]

\[ \liminf_{\tau \to \infty} I_\rho(\tau, h_D(\rho)) = C > 0; \quad (1.12) \]

Remark 2. A surface is said to be strictly convex if its Gaussian curvature is everywhere positive. Therefore, if the obstacle \( D \) is strictly convex, then Theorem 1.1 provides a reconstruction scheme of the shape of \( D \).

1.3. A key integral equality.

Lemma 1.2. Assume \((E, H)\) is a solution of (1.1) or (1.3) satisfying the boundary condition

\[ \nu \wedge H|_{\partial D} = 0 \quad \text{and} \quad \nu \wedge E|_{\partial \Omega} = \nu \wedge E_0|_{\partial \Omega}. \]

We have

\[ I := i\omega \int_{\partial \Omega} (\nu \wedge E_0) \cdot \left[ (\nu \wedge H - \nu \wedge H_0) \wedge \nu \right] dS = i\omega \int_{\partial \Omega} (\nu \wedge E_0) \cdot (H - H_0) dS. \]

First by integration by parts, we have

\[ \int_{\Omega \setminus D} \mu_0^{-1} (\nabla \wedge E) \cdot (\nabla \wedge E - \nabla \wedge E_0) - \omega^2 \varepsilon_0 E \cdot (E - E_0) dx \]

\[ = - \left( \int_{\partial \Omega} - \int_{\partial D} \right) (\nu \wedge \mu_0^{-1} (\nabla \wedge E)) \cdot (E - E_0) dS = 0 \]

by the boundary conditions. Adding this to the following equality

\[ I = \int_{\partial \Omega} (\nu \wedge E_0) \cdot (-i\omega H + i\omega H_0) dS \]

\[ = \int_{\Omega \setminus D} -\mu_0^{-1} (\nabla \wedge E_0) \cdot (\nabla \wedge E) + \omega^2 \varepsilon_0 E_0 \cdot \bar{E} dx \]

\[ + \int_{\Omega} \mu_0^{-1} |\nabla \wedge E_0|^2 - \omega^2 \varepsilon_0 |E_0|^2 dx + \int_{\partial D} (\nu \wedge E_0) \cdot (-i\omega H) dS \]

with the last term vanishing due to the zero-boundary condition on the interface,

\[ \int_{\partial D} (\nu \wedge E_0) \cdot (-i\omega H) dS = \int_{\partial D} (\nu \wedge E_0) \cdot (\bar{\nu} (\nu \wedge H) \wedge \nu) dS = 0, \]
we obtain (1.13). ■

Remark 3. Providing zero tangential electric field instead of magnetic field on the interface

\[(\nu \wedge E)_{\partial D} = 0,\]

we have a similar identity

\[-I = \int_{\Omega \setminus D} \mu_0^{-1} |\nabla \wedge (E - E_0)|^2 - \omega^2 \varepsilon_0 |E - E_0|^2 dx + \int_D \mu_0^{-1} |\nabla \wedge E_0|^2 - \omega^2 \varepsilon_0 |E_0|^2 dx.\]

(1.14)

1.4. Proof of the main theorem. First, we show (1.11) by propose an upper bound of the indicator function. Let \(\tilde{E} = E - E_0\) be the reflect solution in \(\Omega \setminus D\). It satisfies

\[
\begin{align*}
\nabla \wedge (\mu_0^{-1} \nabla \wedge \tilde{E}) - \omega^2 \varepsilon_0 \tilde{E} &= 0 \quad \text{in } \Omega \setminus \bar{D}, \\
\nu \wedge \tilde{E} |_{\partial \Omega} &= 0, \\
\nu \wedge (\mu_0^{-1} \nabla \wedge \tilde{E}) |_{\partial D} &= -\nu \wedge H_0 |_{\partial D} \in TH^{-1/2}(\partial D).
\end{align*}
\]

(1.15)

The well-posedness of this boundary value problem shows

\[\|\tilde{E}\|_{H(\nabla \wedge, \Omega \setminus D)} \leq C \|\nu \wedge H_0\|_{\partial D} \|H^{-1/2}(\partial D)^3} \leq C \|E_0\|_{H(\nabla \wedge, D)},\]

where we denote the general constant \(C > 0\). The second inequality is because for \(v \in H(\nabla \wedge, D),\)

\[
\int_{\partial D} (\nu \wedge H_0) \cdot v dx = -\int_D H_0 \cdot \nabla \wedge v + \nabla \wedge H_0 \cdot v dx
\]

\[= \int_D \frac{i}{\omega} \mu_1^{-1} (\nabla \wedge E_0) \cdot (\nabla \wedge v) + i\omega \varepsilon_0 E_0 \cdot v dx.\]

Notice that

\[\|H_0\|_{H(\nabla \wedge, D)} \leq C \|E_0\|_{H(\nabla \wedge, D)} \leq C (\|E_0\|_{L^2(D)}^3 + \|H_0\|_{L^2(D)}^3) \leq C \|H_0\|_{H(\nabla \wedge, D)}.\]

Therefore, (1.13) implies

\[I_\rho(\tau, t) \leq C (\|E_0\|_{L^2(D)}^3 + \|H_0\|_{L^2(D)}^3).\]

(1.16)

Plug in the CGO-solution (1.10), we obtain the following estimates:

\[\|E_0\|_{L^2(D)}^3 \leq C e^{2\tau(h_D(\rho) - t)\|\eta + R\|_{L^2(D)}}^3 \sim e^{2\tau(h_D(\rho) - t)} \quad \tau \gg 1,\]

\[\|H_0\|_{L^2(D)}^3 \leq C e^{2\tau(h_D(\rho) - t)\|\theta + Q\|_{L^2(D)}}^3 \sim \tau^2 e^{2\tau(h_D(\rho) - t)} \quad \tau \gg 1.\]

Therefore, we obtain

\[I_\rho(\tau, t) \leq C \tau e^{2\tau(h_D(\rho) - t)}\]

for \(\tau\) large enough, proving the first limit (1.11).

To show the second limit (1.12), it suffices to show the following two lemmas.
Lemma 1.3. If $t = h_D(\rho)$ in CGO-solution (1.10), then
\[
\liminf_{\tau \to \infty} \int_D |\nabla \wedge E_0|^2 \, dx = C,
\]
with some constant $C > 0$.

Lemma 1.4. If $t = h_D(\rho)$, then there exists a positive number $c$ such that
\[
\frac{\omega^2 \varepsilon_0 \left( \int_{\Omega \setminus \bar{D}} |E - E_0|^2 \, dx + \int_D |E_0|^2 \, dx \right)}{\mu_0^{-1} \int_D |\nabla \wedge E_0|^2 \, dx} \leq c < 1,
\]
for $\tau$ large enough.

Proof of Lemma 1.3: This is the same proof as in the conductivity case by noticing the left hand side integral
\[
\int_D |\nabla \wedge E_0|^2 \, dx \geq C \|H_0\|_{L^2(D)^3}^2 \geq C \int_D \tau^2 e^{2\tau(x \cdot \rho - h_D(\rho))} \, dx.
\]

Proof of Lemma 1.4: The proof here is essentially the same as for the Helmholtz equation. It suffices to show
\[
\lim_{\tau \to \infty} \frac{\omega^2 \varepsilon_0 \int_{\Omega \setminus \bar{D}} |E - E_0|^2 \, dx}{\mu_0^{-1} \int_D |\nabla \wedge E_0|^2 \, dx} = \lim_{\tau \to \infty} \frac{\varepsilon_0 \int_{\Omega \setminus \bar{D}} |E - E_0|^2 \, dx}{\mu_0 \int_D |H_0|^2 \, dx} = 0.
\]
So we estimate the numerator. Consider the BVP
\[
\begin{cases}
\nabla \wedge p = i\omega \mu_0 q, & \nabla \wedge q = -i\omega \varepsilon_0 p - \frac{i}{\mu_0} \hat{E} \quad \text{in} \quad \Omega \setminus \bar{D}, \\
p |_{\partial \Omega} = 0, & q |_{\partial D} = 0.
\end{cases}
\] (1.17)

******or $\nu \wedge q |_{\partial D} = 0$? which boundary conditions can guarantee $p \in H^2(\Omega \setminus \bar{D})$?

Assume we propose a proper zero boundary condition for this BVP such that it is well-posed for $\hat{E} \in H(\nabla \wedge, \Omega \setminus \bar{D})$, i.e., there exists $p \in H^2(\Omega \setminus \bar{D})^3$, s.t.,
\[
\|p\|_{H^2(\Omega \setminus \bar{D})^3} \leq C \|\hat{E}\|_{L^2(\Omega \setminus \bar{D})^3}.
\]
By the Sobolev embedding, we have
\[
|p(x) - p(y)| \leq C |x - y|^{1/2} \|\hat{E}\|_{L^2(\Omega \setminus \bar{D})^3} \quad \text{for} \quad x, y \in \Omega \setminus \bar{D},
\]
\[
\sup_{x \in \Omega \setminus \bar{D}} |p(x)| \leq C \|\hat{E}\|_{L^2(\Omega \setminus \bar{D})^3}.
\]
Notice that
\[
\nabla \wedge (\mu_0^{-1} \nabla \wedge p) - \omega^2 \varepsilon_0 p = \hat{E}.
\]
Then, integration by parts shows
\[
\int_{\Omega \setminus D} |\tilde{E}|^2 \, dx = \int_{\Omega \setminus D} \tilde{E} \cdot (\nabla \wedge (\mu_0^{-1} \nabla \wedge p) - \omega^2 \varepsilon_0 p) \, dx
\]
\[
= \int_{\Omega \setminus D} \mu_0^{-1} (\nabla \wedge \tilde{E}) \cdot (\nabla \wedge p) - \omega^2 \varepsilon_0 \tilde{E} \cdot p \, dx
\]
\[
+ \left( \int_{\partial \Omega} - \int_{\partial D} \right) \tilde{E} \cdot (\nu \wedge (\mu_0^{-1} \nabla \wedge p)) \, dS
\]
\[
= \int_{\Omega \setminus D} \nabla \wedge (\mu_0^{-1} \nabla \wedge \tilde{E}) \cdot p - \omega^2 \varepsilon_0 \tilde{E} \cdot p \, dx
\]
\[
- \left( \int_{\partial \Omega} - \int_{\partial D} \right) \nu \wedge (\mu_0^{-1} \nabla \wedge \tilde{E}) \cdot p \, dS
\]
\[
= - \int_{\partial D} \nu \wedge (\mu_0^{-1} \nabla \wedge \tilde{E}) \cdot pdS.
\]
Denote by \( x_0 \) the point in \( \{ x \in \partial D \mid x \cdot \rho = h_D(\rho) \} \). We have
\[
\| \tilde{E} \|^2_{L^2(\Omega \setminus D)^3} = \int_{\partial D} (p(x_0) - p(x)) \cdot \nu \wedge (\mu_0^{-1} \nabla \wedge E_0) \, dS - \int_{D} \omega^2 \varepsilon_0 p(x_0) \cdot E_0 \, dx
\]
\[
\leq C \left\{ \int_{\partial D} |x - x_0|^{1/2} |\nu \wedge H_0| \, dS + \int_{D} |E_0| \, dx \right\} \| \tilde{E} \|^2_{L^2(\Omega \setminus D)^3}
\]
\[
\leq C \left\{ \int_{\partial D} \tau |x - x_0|^{1/2} e^{\tau(x_0 - h_D(\rho))} \, dS + \int_{D} e^{\tau(x_0 - h_D(\rho))} \, dx \right\} \| \tilde{E} \|^2_{L^2(\Omega \setminus D)^3}
\]
This yields
\[
\int_{\Omega \setminus D} |\tilde{E}|^2 \, dx \leq C \left\{ \tau^2 \left( \int_{\partial D} |x - x_0|^{1/2} e^{\tau(x_0 - h_D(\rho))} \, dS \right)^2 + \left( \int_{D} e^{\tau(x_0 - h_D(\rho))} \, dx \right)^2 \right\}.
\]
Then follow the step in Helmholtz case to show
\[
\lim_{\tau \to \infty} \tau \int_{\partial D} |x - x_0|^{1/2} e^{\tau(x_0 - h_D(\rho))} \, dS = 0,
\]
where the assumption of the Gaussian curvature is required.

REFERENCES


